

- DEAs / NEAs
- Closure properties (subset construction)
- Regular expressions
  - ↪ automata  $\leftrightarrow$  Reg. expr.
- Pumping lemma
  - Example:  $a^n b^n$
  - Reverse does not hold
- Decision procedures for reg. languages (membership, emptiness, universality, inclusion, equivalence)

membership: linear (DFS)

emptiness:  $\_\ \_\ \_\ \_\ \_\$

universality: PSpace-complete

inclusion:  $\_\ \_\ \_\ \_\ \_\$

Equivalence:  $\_\ \_\ \_\ \_\ \_\$

- logic over finite words (MSO)

- Relations:  $S, \leq, =, Q_a, a \in \Sigma$

- Syntax:  $t := c | x | f(t_1, \dots, t_n) \quad \varphi := t_1 = t_2 \mid P(t_1, \dots, t_n) \quad \psi := \forall t_1 \exists t_2 \varphi_1 \rightarrow \varphi_2 \mid \exists x : p(x)$

- Word model:

$$\omega = (\{0, \dots, n-1\}, S^{\text{int}}, \leq^{\text{int}}, Q_a^{\text{int}})$$

$\forall x : p(x)$

$\exists X : p(X)$

$\forall X : p(X)$

- Simplified MSO: only use second-order variables

$X \in Y$ , Singleton( $X$ ),  $X \subseteq Q_a$ , Succ( $X, Y$ )

- Use bit vectors to encode word models

- Büchi + Elgot's theorem ( $L$  reg.  $\Leftrightarrow$   $L$  MSO-definable)

↪ DFA  $\rightarrow$  MSO (express run  $\exists X_1, \dots, X_n (\dots)$ )

MSO  $\rightarrow$  DFA by induction over structure of formula (uses closure property)

- Complexity: non-elementary (tower of 2)

- Minimization of DFAs

NFA/DFA

- Homomorphisms as merging of states

$O(n^2)$   $O(n \log n)$

- State equivalence: language-equivalent states (Partition refinement algorithm)

- Myhill-Nerode congruence  $\sim_L$

- Canonical DFA  $A_L^*$  (unique minimal up to isomorphism)

- NFA minimization
  - Bisimulation game / Bisimulation equivalence
    - ↳ Game between proposer and responder
  - Bisimulation finer than language equivalence (i.e., Bisimulation implies language equivalence)
  - Marking algorithm  $O(n^3 \cdot m)$ ,  $O(n \log^2 n)$
- Passive learning
  - Samples and consistency
  - Minimal consistent DFA are not unique
  - Passive learning task
  - Prefix tree acceptor
  - Gold's theorem
  - RPNI (merging states of prefix tree acceptor in canonical order)  $O(n^2(n+m))$
  - Learning minimal consistent DFAs using SAT solvers (explicit/implicit encoding)
- Active learning
  - Teacher (membership + equivalence queries)
  - Angluin's algorithm
    - closed and consistent observation table
    - Deriving a DFA from a closed + consistent table
    - Complexity  $O(n)$  equivalence queries +  $O(mn^2)$  membership queries
- Büchi automata
  - Closure properties (closed under union and disjunction)
  - DBW are weaker than NBW ("only finitely many  $a$ ")
    - ↳ DBW are not closed under complement
  - NBW are closed under complement (using eq. similar to Nerode + Ramsey's theorem)
  - $\lim U, U^\omega, \omega\text{-regular expressions}$   $\stackrel{O(n^2)}{\longrightarrow}$  almost op
- Co-Büchi automata
  - $NCW \subseteq NBW$
  - $NCW = DCW$
- Muller automata ( $NM\omega \subseteq NBW, NBW \subseteq NM\omega$ )
- Rabin automata ( $PT = \{(G_i, R_i), \dots, (G_k, R_k)\}, \exists i : \text{Inf}(P) \cap G_i \neq \emptyset \wedge \text{Inf}(P) \cap R_i = \emptyset$ )
  - $NBW \subseteq NRW, NRW \subseteq NBW$
  - $NCW \subseteq NRW$

- Streett automata ( $\inf(\rho) \cap G \neq \emptyset \Rightarrow \inf(\rho) \cap D_i \neq \emptyset$ )
  - $NBW \leq NSW$ ,  $NSW \leq NBW$
- Determinization of Büchi automata
  - Safra's construction ( $NBW \rightarrow DRW$ ),  $2^{\Theta(n \log n)}$  states (optimal)
    - ↳ Uses Safra trees as a compressed version of run trees
- Büchi's theorem
 

SIS over infinite words corresponds to  $\omega$ -regular languages.
- LTL
  - Fix set of AP :  $\varphi := p \in AP \mid \varphi_1 \vee \varphi_2 \mid \neg \varphi \mid X\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Box \varphi \mid F\varphi$
  - Kamp's theorem: LTL is equivalent to FO of one successor
  - Construction of equivalent Büchi automaton via tableau method
    - ↳  $\varphi$ -expansion
    - ↑ (generalized)
    - $2^{\text{acl}(\varphi)}$  states /  $\text{ll}$  acceptance sets
  - There exists a family of LTL formulas s.t. every equivalent Büchi autom. has  $\geq 2^n$  states.
- Church's synthesis problem
- Infinite games
  - ↳ Game graphs, plays, strategies, winning regions, determinacy, positional strategies
- Reachability + Safety games
  - ↳ Attractor / Fixed-point computation
  - Attractor strategy
- Büchi games
  - Recurrence sets
- Parity games
  - Positional determinacy
  - Problem of determine winner is in  $NP \cap coNP$
  - Merging lemma (both players have uniform pos. winning strategies)

- Strategy automata (i.e., Mealy machines)
- Game reduction ( $G \leq G'$ )
  - ↳ If  $G \leq G'$  and Player 0 has winning strategy in  $G'$ , then Player 0 has automaton winning strategy in  $G$ .
- Muller games
  - LARs, upd
  - Muller  $\rightarrow$  parity automata ( $n \cdot n!$  states)
  - Theorem of Büchi + Landweber (Muller games are effectively solvable)
    - ↳ Game reduction using LARs
    - DJW game (any winning strategy automaton has  $\geq n!$  states)
- Nondeterminacy of infinite games
  - Hamming distance
  - infinite XOR function
    - ↳ existence proof using axiom of choice
  - Gale - Stewart game  $G_f$
  - Proof that none of the players has a winning strategy