Advanced Automata Theory (Summer 2017)	Final Exam

This exam has two parts. Please attempt *all* problems from Part I. Part II has four problems and *you can choose any three.*

Part I: Short Questions (25 points)

Attempt all questions in this part.

Problem 1. (1 points) What was the most interesting thing you learnt in this course?

Problem 2. ($6 \times 4 = 24$ points) For each problem, give a brief justification of your answer. In each case, the justification should only be a few lines.

a What is wrong with the proof that $\{0^n 1^n \mid n \geq 0\}$ is regular? For any fixed n, the singleton language $\{0^n 1^n\}$ is regular, as it is finite. Now, since regular languages are closed under union, the union of all these regular languages is also regular.

b Consider the language

$$L = \{u \in \Sigma^* \mid |u|_a > |u|_b\}$$

over the alphabet $\Sigma = \{a, b, c\}$.

Show that L is **not** regular by showing that there exist infinitely many L-equivalence classes. For any two equivalence classes you specify, show that they are, in fact, different.

c Show an example of a non-deterministic Büchi automaton for which the subset construction does not preserve the accepted language.

d Give an linear temporal logic formula for the monadic second order logic formula

$$\forall x. Q_a(x) \rightarrow \exists y. x < y \wedge Q_b(y)$$

e The symmetric difference $L_1 \Delta L_2$ of L_1 and L_2 is the language of all strings in one of the languages but not in the other. Show ω-regular languages are closed under symmetric difference.

f Give an example of two pointed labeled transitions systems and a pair (p, q) of states such that p and q are language equivalent but not bisimilar.

Part II: Long Questions (75 points)

Attempt *any three* questions in this part.

Problem 3. (25 points) An *E-automaton* is a deterministic ω-automaton $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ where $F \subseteq Q$ is a set of final states. We say that $\mathcal{A}$ *E-accepts* a word $\alpha \in \Sigma^\omega$ if and only if the unique run ρ of $\mathcal{A}$ on α satisfies

$$\exists i \in \mathbb{N} \colon \rho(i) \in F.$$

The language accepted by an E-automaton $\mathcal{A}$ is $L(\mathcal{A}) = \{\alpha \in \Sigma^\omega \mid \mathcal{A} \text{ E-accepts } \alpha\}$. A language $L \subseteq \Sigma^\omega$ is called *E-recognizable* if there exists an E-automaton $\mathcal{A}$ with $L = L(\mathcal{A})$.

1. (7 points) Let $\Sigma = \{a, b, c\}$. Give a graphical representation of an E-automaton accepting
$$L = \Sigma^* ac \Sigma^\omega.$$
2. (9 points) Show that
$$\text{an } \omega\text{-language } L \subseteq \Sigma^\omega \text{ is E-recognizable } \Leftrightarrow\ L = U \cdot \Sigma^\omega \text{ for a regular } U \subseteq \Sigma^*.$$
3. (9 points) Show there is a language recognized by a deterministic Büchi automaton which is not E-recognizable.

Problem 4. (25 points) (a) (15 points) Let $G = (V_0, V_1, E)$ be a game graph and $F_1, F_2 \subseteq V$. We consider a game $\mathcal{G} = (G, F_1, F_2)$ with the winning condition

$$\rho \in \text{Win} \ \Leftrightarrow\ \rho \text{ eventually visits } F_1 \text{ and then } F_2 \text{ exactly in this order.}$$

In other words, Player 0 wins a play if he first visits F_1 and then F_2 (but never F_2 before F_1).

Define a game reduction to a classical reachability game $\mathcal{G}' = (G', F')$ with game graph $G' = (V_0', V_1', E')$ by providing

- a suitable memory structure S including an initialization function g;
- the game graph G'; and
- the winning condition F' of $\mathcal{G}'$.

Give a short proof as to why your game reduction is correct.

(b) (10 points) Let G be a game graph and φ_1 and φ_2 two winning conditions (given, e.g., in LTL). If player 1 wins the game with winning condition φ_1 and the game with winning condition φ_2, does she also win the game with winning condition $\varphi_1 \wedge \varphi_2$? If true, write a short proof. If not, show a counterexample.

Problem 5. (25 points) Let Σ be an alphabet. Let $A = (Q, \rightarrow, q_0)$ be an automaton over Σ and let $F \subseteq Q$ be a set of states. A *universal Co-Büchi automaton* (A, F) accepts an infinite word $w \in \Sigma^\omega$ if for *all* runs ρ of A on w, we have that $\mathit{Inf}(\rho) \cap F = \emptyset$. The set of all words accepted by a universal Co-Büchi automaton (A, F) is written $L_{uc}(A, F)$.

1. (10 points) Let $\Sigma = \{a, b\}$ and assume $A = (\{q_0, q_1\}, \{(q_0, a, q_0), (q_0, b, q_0), (q_0, b, q_1), (q_1, b, q_1)\}, q_0)$ and $F = \{q_1\}$. Give an ω-regular expression describing $L_{uc}(A, F)$.

2. (15 points) Prove: $L \subseteq \Sigma^\omega$ is ω-regular if and only if it is accepted by a universal Co-Büchi automaton.

Problem 6. (25 points) The *perfect shuffle* of L_1 and L_2 is the language

$$L_{PS} = \{a_0 b_0 a_1 b_1 \ldots \mid a_i, b_i \in \Sigma \text{ for all } i, a_0 a_1 \ldots \in L_1 \text{ and } b_0 b_1 \ldots \in L_2\}.$$

1. (5 points) What is the perfect shuffle of 0^ω and 1^ω?

2. (20 points) Show that ω-regular languages are closed under perfect shuffle by constructing a Büchi automata for perfect shuffle, starting with Büchi automata for L_1 and L_2. Make sure your automaton is deterministic if the two starting automata are deterministic.