**Advanced Automata Theory (Summer 2017) Problem Set #6 (Due 12.07.2017)**

**Recall:** For a play $\rho = v_0 v_1 \ldots$, we defined

$$Occ(\rho) = \{v \in V \mid v = v_i \text{ for some } i \in \mathbb{N}\}$$

to be the set of vertices occurring in $\rho$.

**Problem 1.** Let $\varphi$, $\psi$ be LTL formulas. We define the *release operator* $R$ as follows:

> *"$\varphi\, R\, \psi$ holds if and only if $\psi$ holds until and including the first point of time where $\varphi$ holds" ($\varphi$ "releases" $\psi$).*

More formally, we define

$$\begin{aligned} \alpha, i \models \varphi\, R\, \psi \;\Leftrightarrow\; & \alpha, j \models \psi \text{ for all } j \geq i, \text{ or} \\ & \exists k \geq i \colon \alpha, k \models \varphi \text{ and } \forall i \leq j \leq k \colon \alpha, j \models \psi \end{aligned}$$

Show the following equivalence: $\varphi\, R\, \psi \equiv \neg(\neg\varphi\, U\, \neg\psi)$.

**Problem 2.** Given is a family $(\varphi_k)_{k \in \mathbb{N}}$ of LTL formulas that are defined as

$$\varphi_k \coloneqq G(p \leftrightarrow X^k\, p) \text{ where } X^k\, p \coloneqq \underbrace{X \ldots X}_{k}\, p.$$

Prove that every Büchi automaton $\mathcal{A}$ with $L(\mathcal{A}) = L(\varphi_k)$ has at least $2^k$ states.

**Problem 3.** Let $\mathbb{B} = \{0, 1\}$.

(a) Give an S1S formula $\varphi_1(X_1, X_2)$ for the language

$$L_1 = \begin{bmatrix}1\\1\end{bmatrix} \begin{bmatrix}1\\0\end{bmatrix}^* \begin{bmatrix}0\\1\end{bmatrix}^\omega \subseteq (\mathbb{B}^2)^\omega.$$

(b) Give an S1S formula $\varphi_2(X_1)$ for the language

$$L_2 = \{\alpha \in \mathbb{B}^\omega \mid \text{the number of zeros occuring in } \alpha \text{ is finite and even}\}.$$

(c) Give an LTL formula $\varphi_3$ for the language $L_1$ from Part (a).

Explain the ideas underlying your formulas.

**Problem 4.** We call a relation $R \subseteq \mathbb{N} \times \mathbb{N}$ *S1S-definable* if there exists an S1S-formula $\varphi(x, y)$ such that

$$(m, n) \in R \;\Leftrightarrow\; \varphi[x/m, y/n] \text{ is satisfied}$$

(i.e. the formula $\varphi(x, y)$ evaluates to `true` when the variables $x$ and $y$ are substituted with the values $m$ and $n$).

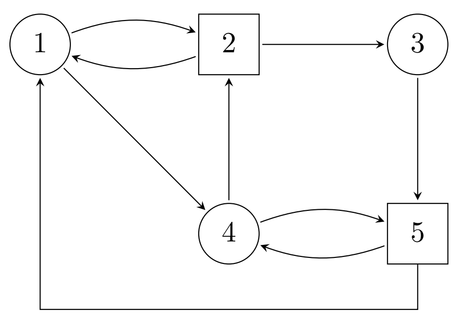
Show that the relation $\mathit{Double} = \{(m, n) \mid n = 2 \cdot m\} \subseteq \mathbb{N} \times \mathbb{N}$ is not S1S-definable.

*Hint: Assume that Double is S1S-definable and, based on this assumption, provide a formula $\varphi$ with*

$$L(\varphi) = \left\{ \begin{bmatrix}0\\0\end{bmatrix}^n \begin{bmatrix}0\\1\end{bmatrix}^n \begin{bmatrix}1\\1\end{bmatrix}^\omega \mid n \geq 1 \right\}.$$

*Then, use a pumping argument to show that $L(\varphi)$ cannot be accepted by a Büchi automaton.*

**Problem 5.** Consider the following game graph.




(a) Compute the winning regions $W_0$ and $W_1$ for the winning condition $|\mathit{Occ}(\rho)| \leq 3$ and give corresponding winning strategies.

(b) Compute the winning regions $W_0$ and $W_1$ for the winning condition $\{2, 5\} \subseteq \mathit{Inf}(\rho)$ and give corresponding winning strategies.

(c) Prove that Player 0 does not have a positional winning strategy in the game of Part (b).

**Problem 6.** A *generalized reachability game* is a tuple $\mathcal{G} = (G, F_1, \dots, F_n)$ consisting of a game graph $G = (V_0, V_1, E)$ and $n$ sets $F_i \subseteq V$. The winning condition of a generalized reachability game is defined by

$$\rho \in \mathit{Win} \;\Leftrightarrow\; \mathit{Occ}(\rho) \cap F_i \neq \emptyset \text{ for each } i \in \{1, \dots, n\}.$$

Show that generalized reachability games can be reduced to reachability games. To this end, provide a reachability game $\mathcal{G}'$ (i.e., find a suitable memory structure $S$, an update function $\delta\colon S \times V \to S$, and an appropriate set $F \subseteq S \times V$) and show that Player 0 wins a play $\rho$ in $\mathcal{G}$ if and only if Player 0 wins the extended play $\rho'$ in $\mathcal{G}'$.

**Problem 7.** Construct a deterministic parity automaton over $\Sigma = \{a, b\}$ for the language

"$a$ occurs infinitely often $\Leftrightarrow$ $b$ occurs finitely often".

**Problem 8.** Consider the Muller automaton depicted below with $\mathcal{F} = \{\{0\}, \{1, 2\}, \{0, 1, 2, 3\}\}$. Give the sequence of LARs as well as their color for the input word $\alpha = ababaabab^\omega$.