We must tweak the setup given in class: Closed expressions and values have no free type variables and $V[\forall\alpha.\tau]\rho$ quantifies over closed types. (Thus, $\delta v = v$ if $v \in \text{CVal}$.)

The model:

$\text{CTyp} := \{ \sigma \in \text{Type} \mid \text{ftv}(\sigma) = \emptyset \}$
$\text{CTerm} := \{ e \in \text{Term} \mid \text{fv}(e) = \emptyset \wedge \text{ftv}(e) = \emptyset \}$
$\text{CVal} := \{ v \in \text{Value} \mid \text{fv}(v) = \emptyset \wedge \text{ftv}(v) = \emptyset \}$

$\text{Cand} := \text{Sub}(\text{CVal})$

$\text{CTerm} \supseteq E[\tau]\rho := \{ e \mid \exists v.\ e \downarrow v \wedge v \in V[\tau]\rho \}$
$\text{CVal} \supseteq V[\tau]\rho$
$V[\alpha]\rho := \rho(\alpha)$
$V[\forall\alpha.\tau]\rho := \{ \Lambda\alpha.e \mid$
$\forall\sigma \in \text{CTyp}.\ \forall S \in \text{Cand}.\ e[\sigma/\alpha] \in E[\tau](\rho, \alpha \mapsto S) \}$
$V[\sigma\to\tau]\rho := \{ \lambda x.e \mid \forall v \in V[\sigma]\rho.\ e[v/x] \in E[\tau]\rho \}$

$D[\Delta] := \{ \rho \in \text{Tyvar} \rightharpoonup \text{Cand} \mid \Delta \subseteq \text{dom}(\rho) \}$
$G[\Gamma]\rho := \{ \gamma \in \text{Var} \rightharpoonup \text{CVal} \mid \forall(x{:}\tau) \in \Gamma.\ \gamma(x) \in V[\tau]\rho \}$

Lemma 1 (LR value inclusion):
If $\text{ftv}(\tau) \subseteq \Delta$
and $\rho \in D[\Delta]$,
then $V[\tau]\rho \subseteq E[\tau]\rho$.

Proof: Omitted.

Lemma 2 (Closure under expansion):
If $e' \mapsto^* e$
and $e \in E[\tau]\rho$,
then $e' \in E[\tau]\rho$.

Proof: Omitted.

Lemma 3 (Validity):
If $\text{ftv}(\tau) \subseteq \Delta$
and $\rho \in D[\Delta]$,
then $V[\tau]\rho \in \text{Cand}$.

Proof: Omitted.

Lemma 4 (Irrelevance):

If $\mathrm{ftv}(\tau) \subseteq \Delta$
and $\rho, \rho' \in D[\Delta]$
and $\forall\alpha \in \mathrm{ftv}(\tau).\ \rho(\alpha) = \rho'(\alpha)$,
then $V[\tau]\rho = V[\tau]\rho'$.

Proof: Omitted.

Lemma 5 (LR substitution):
If $\mathrm{ftv}(\sigma) \subseteq \Delta$
and $\alpha \notin \Delta$,
and $\mathrm{ftv}(\tau) \subseteq \{\alpha\} \cup \Delta$
then $\forall\rho \in D[\Delta]$.
i. $(E[\tau](\rho, \alpha \mapsto V[\sigma]\rho) = E[\tau[\sigma/\alpha]]\rho \wedge$
ii. $V[\tau](\rho, \alpha \mapsto V[\sigma]\rho) = V[\tau[\sigma/\alpha]]\rho)$.

Proof: By induction on τ.

We first note that (i) and (ii) are well-formed. By Lemma 3, we have $V[\sigma]\rho \in \mathrm{Cand}$. Moreover, since $\alpha \notin \Delta \supseteq \mathrm{ftv}(\sigma)$, we have $\mathrm{ftv}(\tau[\sigma/\alpha]) \subseteq \Delta \subseteq \mathrm{dom}(\rho)$.

Set $\rho' := (\rho, \alpha \mapsto V[\sigma]\rho)$.

By the following calculation, we have (ii) $\Longrightarrow$ (i).
$E[\tau]\rho'$
= (Definition of $E[-]-$.)
$\{ e \mid \exists v.\ e \downarrow v \wedge v \in V[\tau]\rho' \}$
= (ii)
$\{ e \mid \exists v.\ e \downarrow v \wedge v \in V[\tau[\sigma/\alpha]]\rho \}$
= (Definition of $E[-]-$.)
$E[\tau[\sigma/\alpha]]\rho$.

To show (ii), we distinguish cases.

Case $\tau = \alpha$.
We have $V[\alpha]\rho' = \rho'(\alpha) = V[\sigma]\rho = V[\alpha[\sigma/\alpha]]\rho$.

Case $\tau = \beta \neq \alpha$.
We have $V[\beta]\rho' = \rho'(\beta) = \rho(\beta) = V[\beta]\rho = V[\beta[\sigma/\alpha]]\rho$.

Case $\tau = \sigma' \longrightarrow \tau'$.
We have
$V[\sigma' \longrightarrow \tau']\rho'$
= (Definition of V at arrow types.)

$\{ \lambda x.e \mid \forall v \in V[\sigma']\rho'.\ e[v/x] \in E[\tau']\rho' \}$

= (IH applied to σ' and ρ.)

$\{ \lambda x.e \mid \forall v \in V[\sigma'[\sigma/\alpha]]\rho.\ e[v/x] \in E[\tau']\rho' \}$

= (IH applied to τ' and ρ.)

$\{ \lambda x.e \mid \forall v \in V[\sigma'[\sigma/\alpha]]\rho.\ e[v/x] \in E[\tau'[\sigma/\alpha]]\rho \}$

= (Definition of V at arrow types.)

$V[(\sigma'[\sigma/\alpha]) \longrightarrow (\tau'[\sigma/\alpha])]\rho$

= (Substitution.)

$V[(\sigma' \longrightarrow \tau')[\sigma/\alpha]]\rho.$

Case $\tau = \forall\beta.\tau'$.

We have

$V[\forall\beta.\tau']\rho'$

= (Definition of V at universal types.)

$\{ \Lambda\beta.e \mid \forall\sigma' \in CTyp.\ \forall S \in Cand.\ e[\sigma'/\beta] \in E[\tau'](\rho', \beta \mapsto S) \}$

= (Definition of ρ' and $\beta \notin dom(\rho')$.)

$\{ \Lambda\beta.e \mid \forall\sigma' \in CTyp.\ \forall S \in Cand.$
$e[\sigma'/\beta] \in E[\tau']((\rho, \beta \mapsto S), \alpha \mapsto V[\sigma]\rho) \}$

= (Lemma 4)

$\{ \Lambda\beta.e \mid \forall\sigma' \in CTyp.\ \forall S \in Cand.$
$e[\sigma'/\beta] \in E[\tau']((\rho, \beta \mapsto S), \alpha \mapsto V[\sigma](\rho, \beta \mapsto S)) \}$

= (IH applied to τ' and $(\rho, \beta \mapsto S)$.)

$\{ \Lambda\beta.e \mid \forall\sigma' \in CTyp.\ \forall S \in Cand.$
$e[\sigma'/\beta] \in E[\tau'[\sigma/\alpha]](\rho, \beta \mapsto S) \}$

= (Definition of V at universal types.)

$V[\forall\beta.(\tau'[\sigma/\alpha])]\rho$

= (Substitution and $\alpha \neq \beta$.)

$V[(\forall\beta.\tau')[\sigma/\alpha]]\rho.$

Q.E.D.

Fundamental theorem:

If $\Delta; \Gamma \vdash e : \tau$,

then $\forall\rho \in D[\Delta].\ \forall\gamma \in G[\Gamma]\rho.\ \forall\delta : \Delta \longrightarrow CTyp.\ \delta\gamma e \in E[\tau]\rho.$

Proof: By induction on $D :: \Delta; \Gamma \vdash e : \tau$.

Case

$$D = \frac{x{:}\tau \in \Gamma}{\Delta; \Gamma \vdash x : \tau}$$

TS: $\delta\gamma x \in E[\tau]\rho \Longleftarrow$ (Lemma 1) $\delta\gamma x = \gamma x \in V[\tau]\rho \Longleftarrow$
(Definition of $G[\Gamma]\rho$.) $x{:}\tau \in \Gamma$.

Case

$$D = \frac{D' :: \Delta; \Gamma, x{:}\sigma \vdash e : \tau}{\Delta; \Gamma \vdash \lambda x.e : \sigma \to \tau}$$

Assume WLOG that $x \notin dom(\gamma)$.

TS: $\delta\gamma(\lambda x.e) = \lambda x.(\delta\gamma e) \in E[\sigma\to\tau]\rho \Leftarrow$ (Lemma 1) $\lambda x.(\delta\gamma e) \in V[\sigma\to\tau]\rho \Leftrightarrow$ (Definition of $V[\sigma\to\tau]\rho$.) $\forall v \in V[\sigma]\rho.\ (\delta\gamma e)[v/x] \in E[\tau]\rho$.

Let $v \in V[\sigma]\rho$ be given. Set $\gamma' := (\gamma, x \mapsto v)$. By the IH applied to D', ρ, γ', δ, we have $(\delta\gamma e)[v/x] = \delta\gamma' e \in E[\tau]\rho$.

Case

$$D = \frac{D_1 :: \Delta; \Gamma \vdash e_1 : \sigma \to \tau \qquad D_2 :: \Delta; \Gamma \vdash e_2 : \sigma}{\Delta; \Gamma \vdash e_1\ e_2 : \tau}$$

TS: $\delta\gamma(e_1\ e_2) = (\delta\gamma e_1)\ (\delta\gamma e_2) \in E[\tau]\rho$.

By the IH applied to D_1, we have $\delta\gamma e_1 \in E[\sigma\to\tau]\rho$. By the definition of $E[\tau]\rho$, there exists v_1 satisfying

$\delta\gamma e_1 \mapsto^* v_1 \in V[\sigma\to\tau]\rho$.

Similarly, from D_2 we obtain v_2 satisfying

$\delta\gamma e_2 \mapsto^* v_2 \in V[\sigma]\rho$.

By the definition of $V[\sigma\to\tau]\rho$, we have $v_1 = \lambda x.e$ for some e satisfying $\forall v \in V[\sigma]\rho.\ e[v/x] \in E[\tau]\rho$. In particular,

$e[v_2/x] \in E[\tau]\rho$.

Combining these facts, we have

$$(\delta\gamma e_1)\ (\delta\gamma e_2) \mapsto^* (v_1\ e_2) \mapsto^* (v_1\ v_2) \mapsto e[v_2/x] \in E[\tau]\rho.$$

Lemma 2 completes the proof.

Case

$$D = \frac{D' :: \Delta, \alpha; \Gamma \vdash e : \tau}{\Delta; \Gamma \vdash \Lambda\alpha.e : \forall\alpha.\tau}$$

Assume WLOG that $\alpha \notin dom(\rho) \cup dom(\delta)$.

TS: $\delta\gamma(\Lambda\alpha.e) = \Lambda\alpha.(\delta\gamma e) \in E[\forall\alpha.\tau]\rho \Leftarrow$ (Lemma 1) $\Lambda\alpha.(\delta\gamma e) \in V[\forall\alpha.\tau]\rho \Leftrightarrow \forall\sigma \in CTyp.\ \forall S \in Cand.\ (\delta\gamma e)[\sigma/\alpha] \in E[\tau](\rho, \alpha \mapsto S)$.

Let $\sigma \in CTyp$ and $S \in Cand$ be given. Set $\rho' := (\rho, \alpha \mapsto S)$ and $\delta' := (\delta, \alpha \mapsto \sigma)$. By the IH applied to D', ρ', γ, δ', we have $(\delta\gamma e)[\sigma/\alpha] = \delta'\gamma e \in E[\tau]\rho'$.

Case

$$D = \frac{D' :: \Delta, \Gamma \vdash e : \forall\alpha.\tau \qquad ftv(\sigma) \subseteq \Delta}{\Delta; \Gamma \vdash e\ \sigma : \tau[\sigma/\alpha]}$$

Assume WLOG that $\alpha \notin \Delta$.

TS: $\delta\gamma(e\ \sigma) = (\delta\gamma e)\ (\delta\sigma) \in E[\tau[\sigma/\alpha]]\rho$.

By the IH applied to D', we have $\delta\gamma e \in E[\forall\alpha.\tau]\rho$. Thus, there exists v satisfying

$$\delta\gamma e \mapsto_* v \in V[\forall\alpha.\tau]\rho.$$

Since $v \in V[\forall\alpha.\tau]\rho$, we have $v = \Lambda\alpha.e'$ for some e' satisfying

1. $\forall\sigma' \in CTyp.\ \forall S \in Cand.\ e'[\sigma'/\alpha] \in E[\tau](\rho, \alpha \mapsto S)$.

We have

$$(\delta\gamma e)\ (\delta\sigma) \mapsto_* v\ (\delta\sigma) \mapsto e'[(\delta\sigma)/\alpha].$$

By Lemma 2, it suffices to show

$$e'[(\delta\sigma)/\alpha] \in E[\tau[\sigma/\alpha]]\rho.$$

By Lemma 3, $V[\sigma]\rho \in Cand$. Instantiating (1) with $\delta\sigma$ and $V[\sigma]\rho$ and applying Lemma 5, we have

$$e'[(\delta\sigma)/\alpha] \in E[\tau](\rho, \alpha \mapsto V[\sigma]\rho) = E[\tau[\sigma/\alpha]]\rho.$$

Q.E.D.