Parametricity and Modular Reasoning
Homework #1

Instructor: Derek Dreyer

Assigned: Tuesday, 30 October 2012
Due: Tuesday, 6 November 2012

1 Encoding Sum Types

We can encode sum types in System F as follows:

$$
\begin{aligned}
\tau_1 + \tau_2 &\stackrel{\text{def}}{=} \forall\alpha.\ (\tau_1 \to \alpha) \to (\tau_2 \to \alpha) \to \alpha \\
\mathrm{inj}_i &\stackrel{\text{def}}{=} \lambda x.\ \overline{\mathrm{inj}_i(x)} \qquad (i \in \{1,2\}) \\
\overline{\mathrm{inj}_i(v)} &\stackrel{\text{def}}{=} \Lambda\alpha.\lambda k_1.\lambda k_2.\ k_i\, v \qquad (i \in \{1,2\}) \\
\mathrm{case}_\tau\ e\ \mathrm{of}\ \mathrm{inj}_1\ x_1 \Rightarrow e_1 \mid \mathrm{inj}_2\ x_2 \Rightarrow e_2 &\stackrel{\text{def}}{=} e[\tau](\lambda x_1.e_1)(\lambda x_2.e_2)
\end{aligned}
$$

1. State the β-reduction rule for this type and show that the encoding satisfies it.
2. Show that if $\vdash f : \tau_1 + \tau_2$, then $\exists i, v.\ \vdash v : \tau_i \wedge f[\tau_1 + \tau_2]\ \mathrm{inj}_1\ \mathrm{inj}_2 \downarrow \overline{\mathrm{inj}_i(v)}$. Your proof should make use of unary parametricity (Girard-style), as we have presented it in class, but you will probably also need to rely on the syntactic property of type preservation.

2 Compatibility of the Binary Logical Relation

Prove that the binary (Reynolds-style) logical relation we defined in class obeys the following *compatibility* rules:

$$
\frac{x : \tau \in \Gamma}{\Delta; \Gamma \vdash x \approx x : \tau}
\qquad
\frac{\Delta; \Gamma, x : \sigma \vdash e_1 \approx e_2 : \tau}{\Delta; \Gamma \vdash \lambda x.e_1 \approx \lambda x.e_2 : \sigma \to \tau}
\qquad
\frac{\Delta; \Gamma \vdash e_1 \approx e_2 : \sigma \to \tau \quad \Delta; \Gamma \vdash e_1' \approx e_2' : \sigma}{\Delta; \Gamma \vdash e_1\, e_1' \approx e_2\, e_2' : \tau}
$$

$$
\frac{\Delta, \alpha; \Gamma \vdash e_1 \approx e_2 : \tau}{\Delta; \Gamma \vdash \Lambda\alpha.e_1 \approx \Lambda\alpha.e_2 : \forall\alpha.\tau}
\qquad
\frac{\Delta; \Gamma \vdash e_1 \approx e_2 : \forall\alpha.\tau \quad \mathrm{FV}(\sigma) \subseteq \Delta}{\Delta; \Gamma \vdash e_1[\sigma] \approx e_2[\sigma] : \tau[\sigma/\alpha]}
$$